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M-ary Channel Quantization for Sequential Decoding

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ABSTRACT

Sequential decoding with direct quantization of the M orthogonal channel outputs is investigated. R_{comp}, the rate below which sequential decoding computations are manageable, is computed for a memoryless, coherent, gaussian channel. These computations show the direct quantization to be more effective than the list-of-ℓ quantization, when sufficient storage is available in the decoding device. The performance of a continuous output channel can be approached by quantizing each output to only a few bits. The direct quantization of channel outputs also simplifies the cumbersome computations which would result if list-of-ℓ decoding were implemented in a general purpose computer.

Accepted for the Air Force
Franklin C. Hudson
Chief, Lincoln Laboratory Office

M-ary Channel Quantization for Sequential Decoding

An M-ary modulation scheme has been used in conjunction with sequential decoding[1] in the Lincoln Experimental Terminal[2,3] (LET). In the M-ary modulation one out of M orthogonal waveforms is transmitted during one transmission period. The receiver then tries to decide which waveform (message) was transmitted by observing the signal amplitudes corresponding to the M possibilities. When M-ary modulation is used with sequential decoding, the communications capacity (for a given transmitted power) can be increased by either making M large or quantizing the outputs of the M orthogonal channels more precisely. Increasing M implies a more complicated system and requires more bandwidth, since the receiving equipment is essentially M receivers, each one matched to one of the M transmitted orthogonal waveforms.

On the list-of-ℓ quantization system, the M orthogonal channel outputs are compared and the receiver designations stored in a list in decreasing order of output amplitude. Only the highest ℓ channels are recorded in the list. This requires storing ℓ identification numbers, each number corresponding to one channel. The decoder then uses the list position of the channel to decide what was transmitted.

If the individual orthogonal channel output amplitudes were quantized directly with some precision, different channel performance would be obtained. The remainder of this note deals with such an investigation.

The expression for R_{comp} (the rate below which one can decode sequentially without excessive computation requirements) was derived for a directly quantized memoryless channel. The equation was evaluated for coherent reception over a channel with gaussian noise. Comparisons of these results with list-of-ℓ quantization are encouraging. The direct computation has a higher absolute potential and even provides a higher R_{comp} for given storage requirements in some interesting cases. It appears that

one may want to quantize the orthogonal channel outputs to as low as two or three bits and would never want to quantize signals to more than four bits.

The direct quantization has the additional advantage of conceptual simplicity as well as being more attractive to use if sequential decoding is performed in a general purpose digital computer.

Computation of λ_i

At the transmitter end we have a stream of bits coming from an information source. These bits are fed into a shift register which has parity networks connected to it. Several parity bits are computed and added to the source bit stream. Then $\log_2 M$ of the bits are grouped together and the modulator selects and transmits one of M orthogonal signals. Although not necessary, we will assume for illustration purposes that the $\log_2 M$ bits consist of one information bit and $\log_2 M - 1$ parity bits. For example, if $M = 16$, one information bit and three parity bits would be used to select an orthogonal waveform. The results obtained, however, are quite general and can be used for other system configurations.

The receiver assumes that up to a particular time it has decoded the received signals into a correct sequence of bits. It uses these bits in a shift register and parity networks identical to the ones at the transmitter and tries either a one or a zero for the next bit. Two sets of $\log_2 M$ bits are obtained, one for the case when a one was hypothesized for the next digit, and the other when a zero was hypothesized. Now a generalized distance metric λ_i, which is a monotonic function of the probability that the hypothesized digit was transmitted given that the particular channel outputs were received is computed for both hypotheses and the hypothesis with the higher λ_i is assumed tentatively to be the transmitted bit. A running sum of the λ_i is kept which hopefully increases consistently as successive bits are decoded. If the sum decreases sufficiently a mistake might have been made somewhere along the way and the process is backed up and different sequences are tried. This sequential decoding process, due to Fano,[4] searches for a sequence which

will eventually give the highest sum of λ_i.

Let the hypothesized transmitted message be designated by x_i^* (one of the M orthogonal transmitted signals), and let $\vec{y}$ be the vector of outputs of the M orthogonal channels during the i-th transmission of the channel. We compute the a posteriori probability that x_i^* was transmitted given that $\vec{y}_i$ was received. For a sequence of n-1 bits the a posteriori probability of the decoder picking the transmitted sequence of bits is the product of the individual bit a posteriori probabilities, if the channel is memoryless and the noise is white gaussian. The object of the decoder then is to maximize this a posteriori probability or any monotonic function of it. The logarithmic function of this probability allows simple calculations.

$$L_n = \sum_{i=0}^{n-1} [\log_2 p(x_i^* | \vec{y}_i) - U] = \sum_{i=0}^{n-1} \lambda_i \quad , \tag{1}$$

where x_i^* is the i-th hypothesized transmitted symbol, $\vec{y}_i$ is the i-th received vector of channel outputs, L_n is the measure of correctness up to the n-th decision and U is simply a constant selected so that L_n increases for correct sequences of decoded bits and decreases for incorrect sequences. The exact value of U has been shown not to affect the number of decoding computations greatly.[5]

In a list-of-ℓ type demodulation system the outputs of the M orthogonal channels are compared with one another, and a list is kept of the ℓ channel identification numbers with the highest output signals. A table of values of λ (called a metric) is kept which assigns a value for λ to the various channels depending on what position in the list the channel occupied. This method loses some information, since the metric is computed for the average channel conditions. The individual orthogonal channel amplitudes are not utilized, only the relative positions on the list count. The following paragraphs describe how the individual channel output amplitudes can be used in the calculations if the individual channel outputs are quantized directly and stored in the decoding computer.

The λ_i can be computed directly when the individual channel outputs are available. We can compute $p(x_i^* | \vec{y}_i)$ by Bayes rule:

$$p(x_i^* | \vec{y}_i) = \frac{p(x_i^*)\, p(\vec{y}_i | x_i^*)}{p(\vec{y}_i)} \quad . \tag{2}$$

When the k-th orthogonal waveform is transmitted, the k-th orthogonal channel output will be a random variable with an amplitude probability density $p_s(y_k)$ due to transmitted signal and noise, and the other channel outputs will have an amplitude probability density $p_n(y_j)$ due to noise only. These densities are independent of i.

Given that the k-th signal (x_{ik}^*) was transmitted during the i-th interval, we have the conditional probability

$$p(\vec{y}_i | x_{ik}^*) = p_s(y_{ik}) \prod_{\substack{j=1 \\ j \neq k}}^{M} p_n(y_{ij}) \; . \tag{3}$$

Since the individual noise components are statistically independent, the probability that a particular $\vec{y}$ occurs is

$$p(\vec{y}_i) = \sum_{h=1}^{M} p(x_h^*)\, p_s(y_h) \prod_{\substack{j=1 \\ j \neq h}}^{M} p_n(y_i) \; , \tag{4}$$

where $p(x_h^*)$ is the probability that the h-th message was transmitted during an interval and is assumed to be a constant $1/M$. Using Eq. (3) and (4) in Eq. (2) we have

$$p(x_{ik}^* | \vec{y}_i) = \frac{p_s(y_{ik})/p_n(y_{ik})}{\sum_{h=1}^{M} p_s(y_{ih})/p_n(y_{ih})} \quad , \tag{5}$$

where $p(x_{ik}^* | \vec{y}_i)$ is the probability that the k-th message was transmitted during i-th transmission interval given that $\vec{y}_i$ was received. Then the

measure of correctness of the i-th decision is given by

$$\lambda_i = \log_2 [p_s(y_{ik})/p_n(y_{ik})] - \log_2 [\sum_{h=1}^{M} p_s(y_{ih})/p_n(y_{ih})] - U, \quad (6)$$

if the k-th message was hypothesized.

The sequential decoding algorithm requires the computation of λ_i every time the decoder tries to decide on a bit; hence, the computer must be able to compute λ_i very quickly.

If one computed λ_i by the list-of-ℓ technique, one would have to search through the ℓ channel identification numbers, find the position of the desired message and then look up the λ_i which corresponds to that position. While table look up is easy in a general purpose digital computer, finding a particular number in a list is not. Consequently list-of-ℓ decoding is a very cumbersome, time consuming operation on a general purpose digital computer. In addition, getting the list itself is not a trivial matter; indeed, in the LET system the list is formed by analog circuitry[6] external to the special purpose digital decoding machine.

Equation (6) shows the operations required to obtain the metric with direct quantization. Note that the second term of (6) is independent of k, thus has to be computed only once no matter what k is hypothesized. To compute this term, one would look up and sum the ratio of the probabilities and then compute the logarithm of the sum. The frequently computed first term of (6) can be obtained by a simple table look up. The tables for both of these computations can be constructed beforehand, or the statistics $p_s(y_{ik})$ and $p_n(y_{ik})$ can be updated periodically, thus adjusting the metric computations to the most recently observed channel characteristics.

R_{comp} of Quantized System

In order to compute λ_i, we need to have the values of $\vec{y}_i$ stored in some fashion. One way to accomplish this is to quantize the output of each

orthogonal channel and store the quantized values in a digital computer memory. We will investigate the quantization requirements by computing R_{comp} (the rate below which one can sequentially decode without excessive computation requirement) for various quantization step sizes and limits of quantizations.

We need to compute R_{comp} for a discrete, memoryless channel with M channel inputs and J possible channel outputs. The output can be visualized as a vector $\vec{y}$ with M components $(y_1, y_2, y_3, \ldots, y_M)$. Each component is quantized into N intervals so that the channel output $\vec{y}$ can take on N^M different values $(J = N^M)$.

Reiffen[7] has shown that R_{comp} for a discrete, memoryless channel such as ours is given by

$$R_{comp} = \max_{\{p(x_k)\}} \left\{ -\log_2 \sum_{j=1}^{J} \left[\sum_{k=1}^{M} p(x_k) \sqrt{p(\vec{y}_j | x_k)} \right]^2 \right\} , \quad (7)$$

where $p(x_k)$ is the probability of the k-th message (x_k) being transmitted and $p(\vec{y}_j | x_k)$ is the transition probability of $\vec{y}_j$ being received given that x_k was transmitted. The value of the transition probability is given by

$$p(\vec{y}_j | x_k) = p_s(y_k) \prod_{\substack{m=1 \\ m \neq k}}^{M} p_n(y_m) , \quad (8)$$

where $p_s(y_k)$ is the discrete probability density function of the channel output of k-th channel (i.e., the channel containing the signal) and $p_n(y_m)$ is the discrete probability density function of the other channels (i.e., channels with noise only).

Substitution of (8) into (7) yields after simplification and summing probabilities to 1

$$R_{comp} = \max_{\{p(x_k)\}} \left\{ -\log_2 \left(\sum_{k=1}^{M} p^2(x_k) + \left[\sum_{n=1}^{N} \sqrt{p_n(y_n) p_s(y_n)} \right]^2 \sum_{k=1}^{M} \sum_{\substack{g=1 \\ g \neq k}}^{M} p(x_k) p(x_g) \right) \right\} , \quad (9)$$

where $p_s(y_n)$ is the probability that the orthogonal channel output with the signal falls into the n-th quantization interval and $p_n(y_n)$ is the probability that the orthogonal channel with noise only falls into the n-th quantization interval. Equation (9) is seen to be maximized when $p(x_k)$ is a constant $1/M$. Thus

$$R_{comp} = -\log_2 \left\{ \frac{1}{M} + \frac{M-1}{M} \left[\sum_{n=1}^{N} \sqrt{p_n(y_n)\, p_s(y_n)} \right]^2 \right\}. \tag{10}$$

Results

Equation (10) was evaluated for a coherent gaussian channel whose

$$p_s(y) = \frac{1}{\sqrt{2\pi}} e^{\frac{-(y-a)^2}{2}} \tag{11}$$

and

$$p_n(y) = \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} , \tag{12}$$

where $p_s(y)$ and $p_n(y)$ are the continuous probability densities for the signal and noise respectively and $a = \sqrt{\frac{2E_r}{N_o}}$, in which E_r is the received energy when an orthogonal waveform is transmitted and N_o is the single-sided white gaussian noise power density.

The output space y is partitioned into N increments and the values of $p_s(y_n)$ and $p_n(y_n)$ are simply integrals of (11) and (12) over the n-th increment. For all calculations, equal increments were assumed between some minimum value y_{min} and maximum value y_{max}, which correspond to the minimum and maximum channel output voltages. The lower end increment and the upper end increment were made to include all values of y less then y_{min} and all values of y more than y_{max} respectively. The values of y_{min} and y_{max} were optimized so that R_{comp} was maximized. Figures 1-4 show R_{comp}

as a function of E_r/N_o for $N = 2, 4, 8, 16$ and different values of M. The figures also show the continuous case, which would result if the channel outputs were quantized into an infinite number of increments. This is the best that sequential decoding could do with an M-ary channel. The figures show the list-of-ℓ R_{comp} for different values of ℓ* as well.

It is obvious that the direct quantization of signals has a higher absolute potential than the list-of-ℓ quantization. Figure 5 shows signal-to-noise ratios that are required to obtain a typical R_{comp} with a given available storage. One can see that the direct quantizer requires a lower signal-to-noise ratio only when large amount of storage is available. If storage availability is no problem it appears that $N = 8$ or 16 might be as fine a quantization that one would use. An interesting fact is that the direct quantization is more efficient when low values of R_{comp} are required.

The optimum quantization levels are shown in Fig. 6. As shown by Eq. (10) the levels are independent of M, so curves of optimum quantization region are shown as a function of E_r/N_o for values of N. Since the signal and noise distributions are symmetrical, $y_{max} = a - y_{min}$ is the proper choice. The value of y_{min} and y_{max} is seen not to be a very critical one. Figure 7 shows the plots of R_{comp} as a function of the limits, which indicates that the maximum is quite broad and not much is lost at other signal-to-noise ratios if the limiting and quantization are picked optimum for the minimum expected signal-to-noise ratio. The curves, however, assume that the probability densities which are used in the computer are the correct ones. If the probabilities change and the computer does not change the metric calculation tables the quantization interval is somewhat more sensitive to channel variations.

Conclusion

The direct quantization of every orthogonal channel output to a modest number of bits (3 or 4 bits) provides R_{comp} very close to the best that can be accomplished over an M-ary channel. In comparison, this is substantially

*The list-of-ℓ curves are taken from K. Jordan's work (Ref. 5) which has guided the present study in all phases.

better than list-of-ℓ quantization. When the direct quantization is compared to list-of-ℓ quantization assuming the same memory storage, the direct quantization is better for more complicated systems, that is systems with larger memories. The list-of-ℓ provides higher R_{comp} for very limited memory machines. The computation of λ_i for direct quantized system is also easier to perform on the general purpose digital computer.

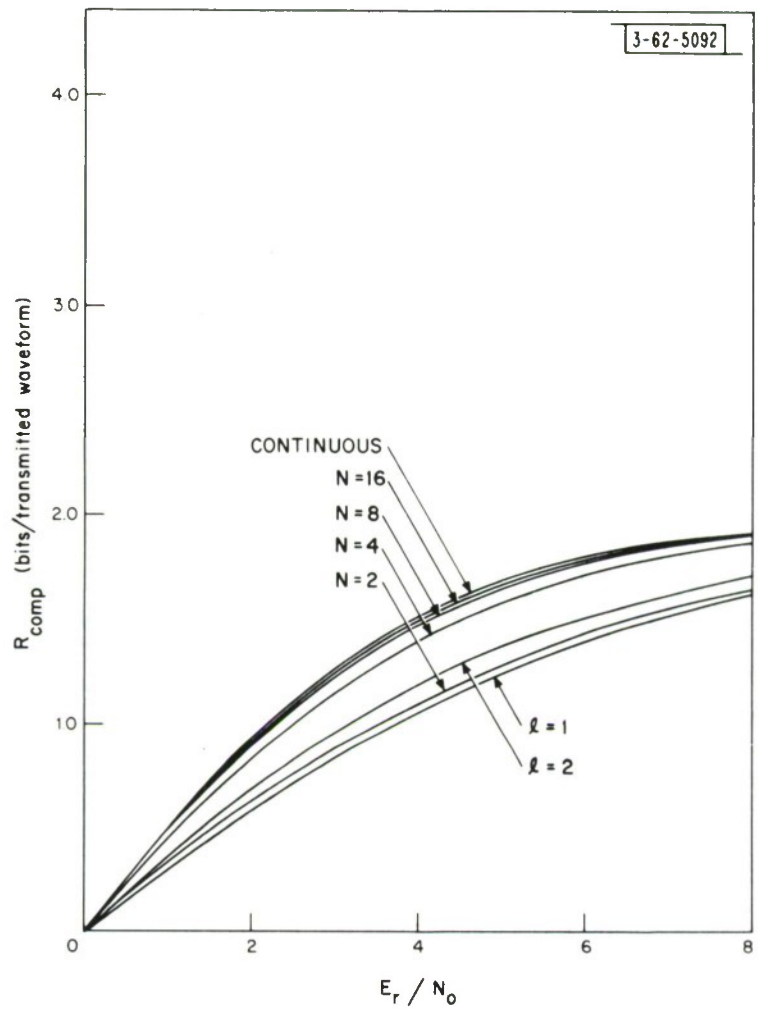

Fig. 1. R_{comp} for M = 4.

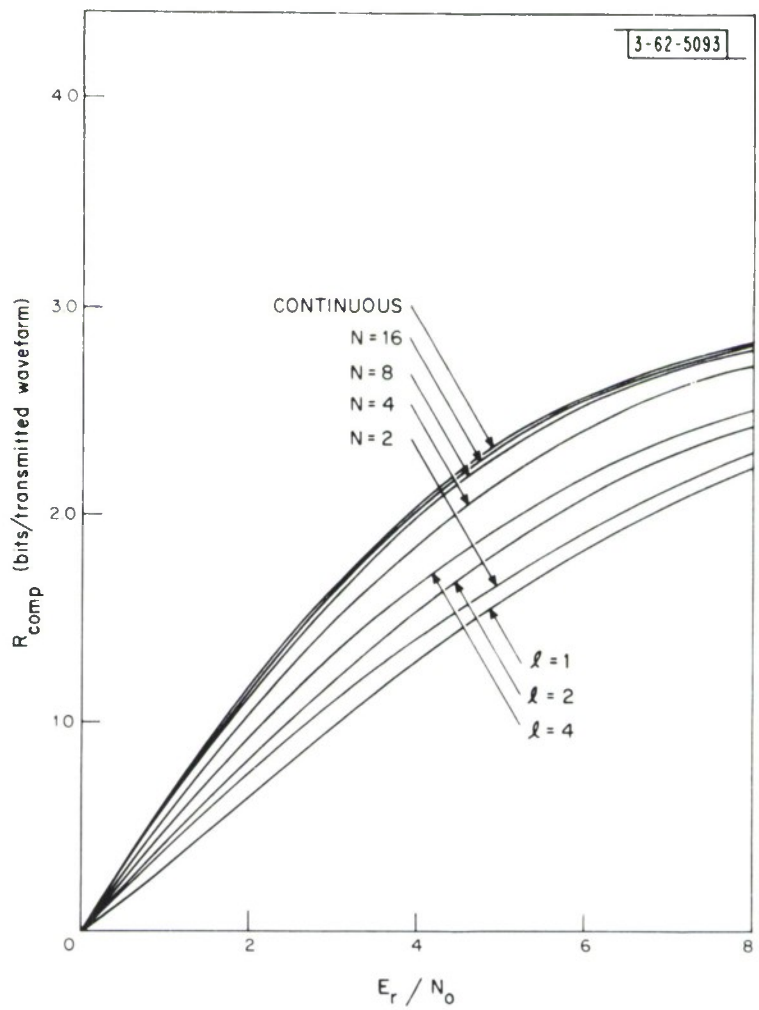

Fig. 2. R_{comp} for M = 8.

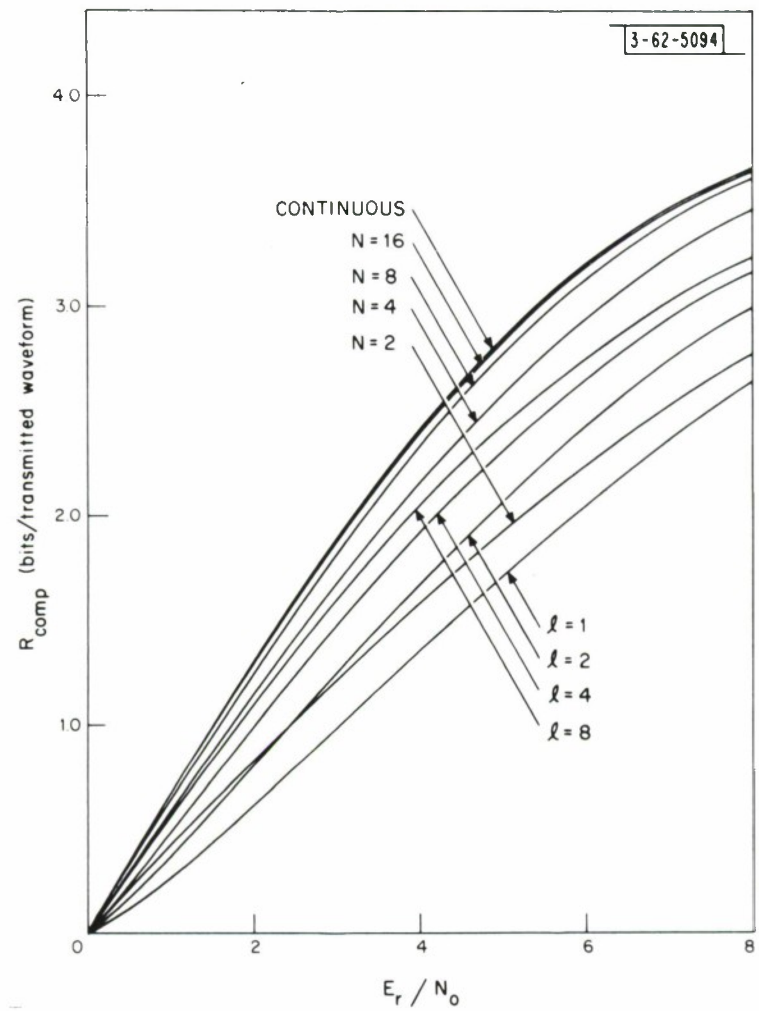

Fig. 3. R_{comp} for M = 16.

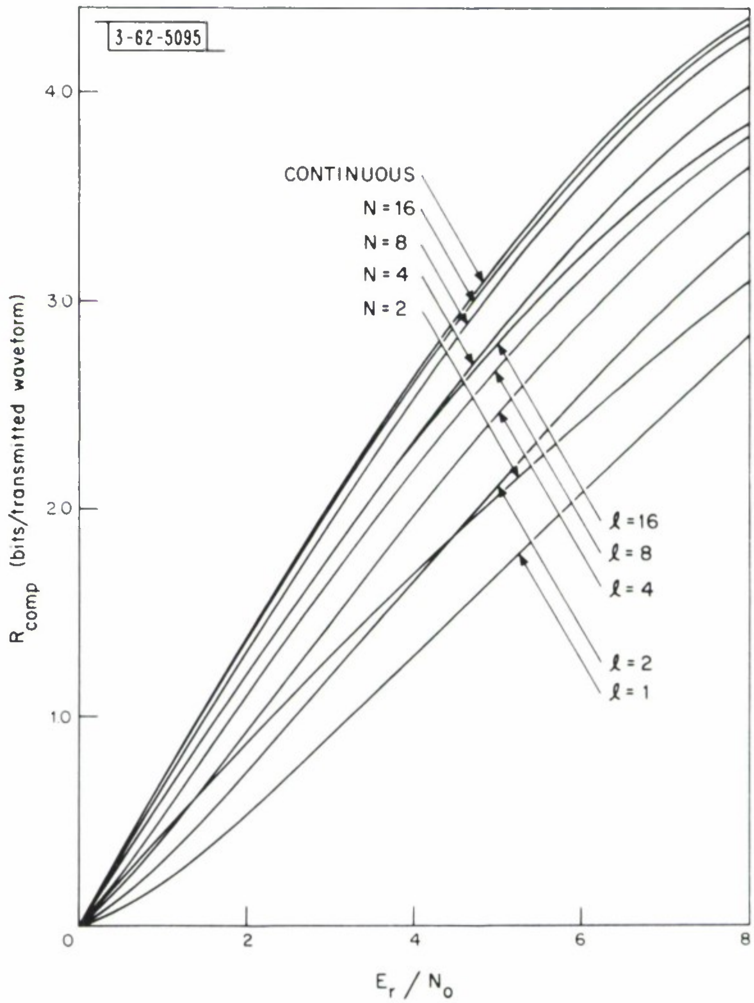

Fig. 4. R_{comp} for M = 32.

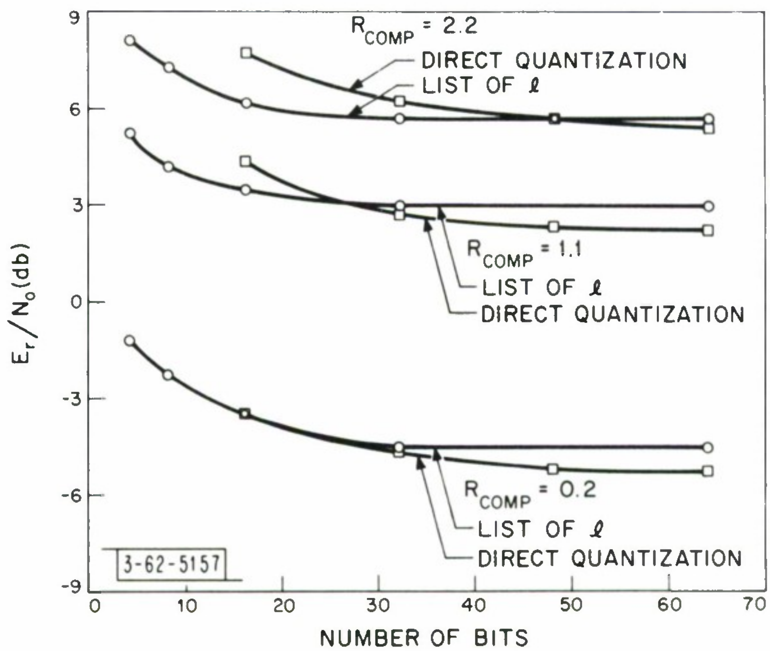

Fig. 5. E_r/N_o required for given R_{comp} as a function of number of storage bits M = 16.

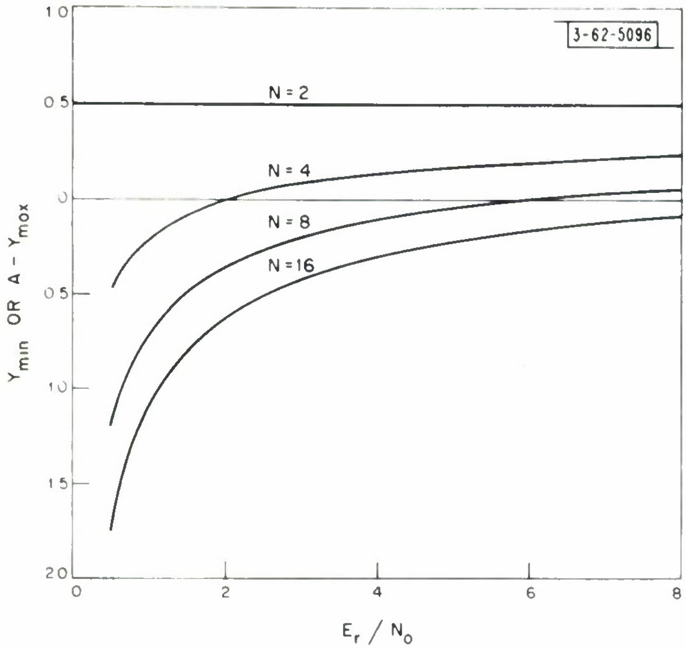

Fig. 6. Optimum quantization limiting.

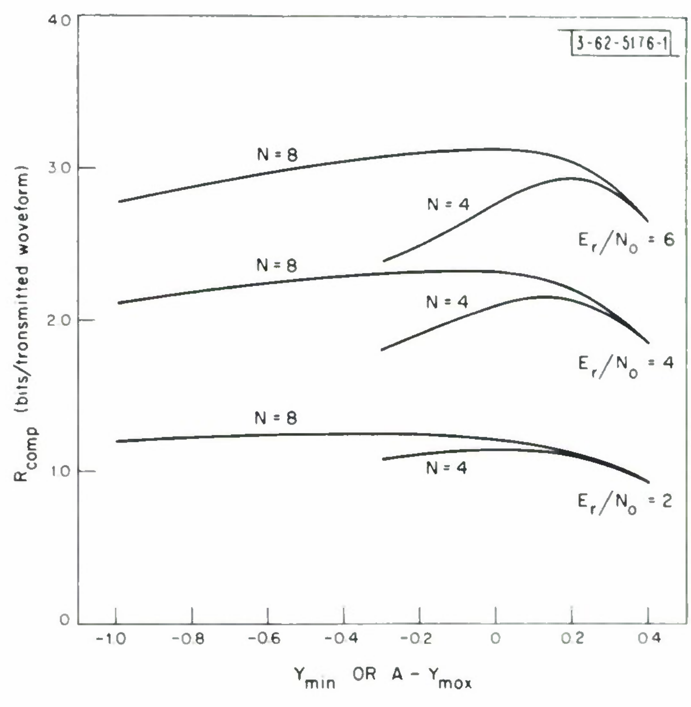

Fig. 7. Effect of limiting on R_{comp}.

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13. ABSTRACT

Sequential decoding with direct quantization of the M orthogonal channel outputs is investigated. R_{comp}, the rate below which sequential decoding computations are manageable, is computed for a memoryless, coherent, gaussian channel. These computations show the direct quantization to be more effective than the list-of-ℓ quantization, when sufficient storage is available in the decoding device. The performance of a continuous output channel can be approached by quantizing each output to only a few bits. The direct quantization of channel outputs also simplifies the cumbersome computations which would result if list-of-ℓ decoding were implemented in a general purpose computer.

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LET	quantization	signal-to-noise ratio

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